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The Spheroidal Method in Satellite Astronomy*

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1. Introduction.

If $\underset{\sim}{r}$ is the position vector of an artificial satellite of an oblate planet, relative to the latter's center of mass, the drag-free motion of the satellite is determined by the differential equation

$$\ddot{\underset{\sim}{r}} = - \nabla V \tag{1}$$

Here the gravitational potential V of the planet is expressible as an expansion in spherical harmonics

$$V = - \frac{\mu}{r} [1 - \sum_{n=2}^{\infty} (\frac{r_e}{r})^n J_n P_n(\sin\theta)] + \text{tesseral harmonics} \tag{2}$$

where $r = |\underset{\sim}{r}|$, θ is the declination, r_e is the equatorial radius, P_n is the n'th Legendre polynomial, and μ = GM, the product of the gravitational constant and the mass of the planet. Besides the drag, Eqs. (1) and (2) neglect the lunar-solar perturbation and all non-gravitational forces. The constants J_n are pure numbers characterizing the

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planet's potential, with $J_2 = (1.08)10^{-3}$ for the Earth and with all the other J_n's of the order 10^{-6} or smaller.

2. Possible Reference Orbits.

Most approaches to the problem of solving (1) and (2) for the orbit have begun with replacing V by $V_o \equiv -\mu/r$ and finding the perturbations of the resulting elliptic orbit, produced by the higher harmonics. Sterne(1957,1958) and Garfinkel(1958,1959) both began with potentials of the form $V = f(r,\theta)$, taking into account part of the effect of the second harmonic. Further progress then depends on finding how the resulting intermediate or reference orbit changes with time.

To take advantage of our knowledge of the actual shape of the earth, or of any oblate planet more closely resembling an oblate spheroid than a sphere, the author (Vinti 1959a, 1959b) decided to try oblate spheroidal coordinates. If X, Y, and Z are the usual rectangular coordinates, these spheroidal coordinates ρ, η, and ϕ are defined by the equations

$$X + iY = r\cos\theta\exp i\phi = [(\rho^2+c^2)(1-\eta^2)]^{\frac{1}{2}}\exp i\phi \tag{3.1}$$

$$Z = r\sin\theta = \rho\eta \tag{3.2}$$

Here c is an adjustable distance, small compared to r_e. For large r, $\rho \to r$ and $\eta \to \sin\theta$. The surfaces ρ = constant are oblate spheroids, approaching sphericity as ρ increases, and the surfaces η = constant are hyperboloids, asymptotic to the cones θ = constant.

With the hope of obtaining a more accurate reference orbit as a starting point for the calculation of satellite orbits, the author wrote out the Hamilton-Jacobi equation in these coordinates, finding that it would be separable if V has the general form

$$V' = (\rho^2 + c^2\eta^2)^{-1}[\, f(\rho) + g(\eta)] \tag{4}$$

On imposing the requirement that V' shall be a solution of Laplace's equation $\nabla^2 V' = 0$ and that this solution shall be non-singular on the Z-axis, one finds that the functions $f(\rho)$ and $g(\eta)$ can only be

$$f(\rho) = b_1\rho \qquad\qquad g(\eta) = b_2\eta \tag{5}$$

Placing the origin at the center of mass then makes $b_2 = 0$ and requiring V' to have the form $-\mu/r$ at large r makes $b_1 = -\mu$. Then

$$V' = -\frac{\mu\rho}{\rho^2 + c^2\eta^2} = -\mu \mathrm{Re}(\rho + ic\eta)^{-1} \tag{6}$$

The expansion of V' in zonal harmonics

$$V' = -\frac{\mu}{r}[\, 1 - \frac{c^2}{r^2} P_2(\sin\theta) + \frac{c^4}{r^4} P_4(\sin\theta) - \frac{c^6}{r^6} P_6(\sin\theta) + \ldots] \tag{7}$$

then shows that V' agrees with V through the second harmonic if

$$c^2 = r_e^2\, J_2 \tag{8}$$

With such a choice for c, we also find that $J_4 = -J_2^2,\ J_6 = J_2^3,\ \ldots$

Since observations show that $J_4 \approx -1.5\, J_2^2$, it follows that V' also represents about two-thirds of the fourth harmonic. It follows that V' accounts for about 99.5% of the departure of V from the simple value $-\mu/r$ that would hold for a spherically symmetric planet. In other words the geoid constructed ~~form~~ from V' never departs from the actual sea-level surface by more than a few hundred feet. Furthermore, Weinacht(1924) proved that ~~the~~ separable motion of a particle in Euclidean space is either a Staeckel system or reducible to a Staeckel system by a point transformation. Of the eleven systems of coordinates in which Staeckel systems may be expressed, the oblate spheroidal has the most appropriate symmetry. Furthermore, Eq.(6) is the most flexible solution of Laplace's equation in this system that leads to separability. It therefore appears likely that the orbit of a particle moving in the potential field (6), with $c^2 = r_e^2\, J_2$, is the best possible reference orbit that can be chosen, from the point of view of accuracy of fit to the actual orbit without perturbation theory.

3. The Quadratures

If α_1 is the energy, α_3 the axial component of angular momentum, and α_2 a separation constant that would reduce to the total angular momentum in the Keplerian case $c = 0$, then with the potential (6) the Hamilton-Jacobi equation separates, with a solution

$$W = W_1(\rho,\alpha_1,\alpha_2,\alpha_3) + W_2(\eta,\alpha_1,\alpha_2,\alpha_3) + \alpha_3\phi \tag{9}$$

If β_1, β_2, and β_3 are constants, such that in the Keplerian case $-\beta_1$ would be the time of passage through perigee, β_2 the argument of perigee, and β_3 the right ascension of the node, the coordinates ρ, η, and ϕ are given by

$$\frac{\partial W}{\partial \alpha_1} = t + \beta_1 = \pm \int_{\rho_1}^{\rho} \rho^2 F^{-\frac{1}{2}} d\rho \pm c^2 \int_{0}^{\eta} \eta^2 G^{-\frac{1}{2}} d\eta \tag{10.1}$$

$$\frac{\partial W}{\partial \alpha_2} = \beta_2 = \mp \alpha_2 \int_{\rho_1}^{\rho} F^{-\frac{1}{2}} d\rho \pm \alpha_2 \int_{0}^{\eta} G^{-\frac{1}{2}} d\eta \tag{10.2}$$

$$\frac{\partial W}{\partial \alpha_3} = \beta_3 = \phi \mp \alpha_3 \int_{0}^{\eta} (1-\eta^2)^{-1} G^{-\frac{1}{2}} d\eta \pm c^2\alpha_3 \int_{\rho_1}^{\rho} (\rho^2+c^2)^{-1} F^{-\frac{1}{2}} d\rho \tag{10.3}$$

Here

$$G(\eta) = -\alpha_3^2 + (1-\eta^2)(\alpha_2^2 + 2\alpha_1 c^2\eta^2) = (\alpha_2^2-\alpha_3^2)\left(1-\frac{\eta^2}{\eta_0^2}\right)\left(1-\frac{\eta^2}{\eta_2^2}\right) \Bigg| \left(-\eta_0 \leqq \eta \leqq \eta_0 \leqq 1\right) \tag{11.1}$$

a quadratic in η^2, and

$$F(\rho) = c^2\alpha_3^2 + (\rho^2+c^2)(-\alpha_2^2+ 2\mu\rho+ 2\alpha_1\rho^2)=(-2\alpha_1)(\rho-\rho_1)(\rho_2-\rho)(\rho^2+A\rho+B),(\rho_1 \leqq \rho \leqq \rho_2)$$

a quartic in ρ . (11.2)

4. Factoring the Quartics [For references on Sects. 4,5, and 6, see Vinti 1961b, 1961c, and 1962]

Finding the coordinates as functions of the time depends on inverting Eqs.(10.1) and (10.2) to obtain ρ and η in terms of t and then inserting the results into (10.3) to obtain ϕ . To do this we must first evaluate the above integrals and this evaluation requires factoring the quartics $F(\rho)$ and $G(\eta)$.

We may define constant orbital elements $a_o \equiv -\mu/2\alpha_1$, $e_o \equiv (1+2\alpha_1\alpha_2^2/\mu^2)^{\frac{1}{2}}$, and $i_o \equiv \cos^{-1}(\alpha_3/\alpha_2)$, β_1,β_2,β_3 that can be obtained directly from initial conditions. In this way we can factor $G(\eta)$ exactly and $F(\rho)$ through order J_2^2 without difficulty. A somewhat better set of elements is a,e,I,β_1, β_2, β_3, introduced by Izsak (1960). Here $a \equiv \frac{1}{2}(\rho_1+\rho_2)$, $e \equiv (\rho_2-\rho_1)/(\rho_2+\rho_1)$, and $I = \sin^{-1}\eta_o$. The quantities $\alpha_1,\alpha_2,\alpha_3$, A,B, and η_2 can all be expressed in terms of these elements, so that the latter lead to an exact factorization of $F(\rho)$. These elements can be obtained from initial conditions by numerical solution of $F(\rho) = 0$, but they can be determined without such a procedure by iterated least-square fitting to an observed orbit.

5. Evaluating the Integrals

The integrals in (10.3) can be expressed as incomplete elliptic integrals of the third kind and those in (10.1) and (10.2) as incomplete elliptic integrals of the first and second kinds. It is simpler, however, to avoid such a formulation. Suppose we introduce the uniformizing variables E, v, ψ, and χ, defined by

$$\rho = a(1-e\cos E) = a(1-e^2)(1 + e\cos v)^{-1} \tag{12.1}$$

$$\eta = \eta_o \sin\psi \tag{12.2}$$

$$\exp i\chi = (1-\eta_o^2\sin^2\psi)^{-\frac{1}{2}}(\cos\psi + i\sqrt{1-\eta_o^2}\sin\psi) \tag{12.3}$$

Here E and v are analogous, respectively, to the eccentric and true anomalies in elliptic motion, ψ to the argument of latitude, and χ to the projection of the orbital arc on the equator. By using an expansion in Legendre polynomials with argument $-\frac{1}{2} AB^{-\frac{1}{2}}$, viz.,

$$(1 + A/\rho + B/\rho^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} (B^{\frac{1}{2}} \rho^{-1})^n P_n(-\tfrac{1}{2} AB^{-\frac{1}{2}}), \tag{13}$$

we can express the ρ-integrals R_1, R_2, R_3, occurring respectively in (10.1), (10.2), and (10.3), in the forms

$$(-2\alpha_1)^{\frac{1}{2}}R_1 = -\tfrac{1}{2} AE + a(E-e\sin E)+ A_1 v + \sum_{j=1}^{2} A_{1j} \sin jv \tag{14.1}$$

$$(-2\alpha_1)^{\frac{1}{2}} R_2 = A_2 v + \sum_{j=1}^{4} A_{2j} \sin jv \tag{14.2}$$

$$(-2\alpha_1)^{\frac{1}{2}} R_3 = A_3 v + \sum_{j=1}^{4} A_{3j} \sin jv \tag{14.3}$$

Here the coefficients A_1, A_2, and A_3 are infinite series, leading to an exact evaluation of secular effects for the reference orbit, and the sine terms are carried far enough to give periodic effects through order J_2^2 .

We can express the corresponding η- integrals as

$$N_1 = C_1\psi + \sum_{j=1}^{2} C_{1j} \sin 2j\psi \tag{15.1}$$

$$N_2 = C_2\psi + \sum_{j=1}^{2} C_{2j} \sin 2j\psi \tag{15.2}$$

$$N_3 = C_3 X + C_4\psi + C_{32}\sin 2\psi \tag{15.3}$$

Here C_1 and C_2 are expressible in terms of the complete elliptic integrals $K(\eta_0/\eta_2)$ and $E(\eta_0/\eta_2)$ and N_3 in terms of an infinite series. The results for the η- integrals are thus also accurate enough to give secular effects exactly for the reference orbit and periodic effects through order J_2^2 .

6. Solution of the Kinetic Equations (10)

One then inserts (14) and (15) into Eqs. (10.1) and (10.2), placing

$$E = M_s + E_p \qquad v = M_s + v_p \qquad \psi = \psi_s + \psi_p \tag{16}$$

The secular terms M_s and ψ_s are then found by dropping E_p, v_p, ψ_p and the sine terms in (10.1) and (10.2) and solving a pair of linear algebraic equations. The secular mean anomaly M_s appears as the product of $2\pi\nu_1$ and a linear function of $t + \beta_1$; the secular term ψ_s is the product of $2\pi\nu_2$ and a linear function of $t + \beta_2$. Here ν_1 and ν_2 are, respectively, the mean ρ-frequency $\partial\alpha_1/\partial j_1$ and the mean η-frequency $\partial\alpha_1/\partial j_2$, j_1 and j_2 being the corresponding action variables(Vinti 1961a).

One then expresses the periodic terms as

$$E_p = E_0 + E_1 + E_2 \qquad v_p = v_0 + v_1 + v_2 \qquad \psi_p = \psi_0 + \psi_1 + \psi_2 \, , \tag{17}$$

where E_n, e.g., denotes a periodic part of order J_2^n. One then places $E_p = E_0$, $v_p = v_0$, $\psi_j = \psi_0$ into (10), rejecting all periodic terms of order J_2 or higher, and solves (10.1), (10.2), and (12.1) for E_0, v_0, and ψ_0. Here $M_s + E_0$ appears as the solution of the Kepler equation

$$M_s + E_0 - e'\sin(M_s + E_0) = M_s \tag{18.1}$$

$$e' \equiv ae/a_0 < e \tag{18.2}$$

One continues by adding in the terms E_1, v_1, and ψ_1 into Eqs.(10), rejecting only those periodic terms of order J_2^2 or higher. Then $M_s + E_0 + E_1$ satisfies a similar Kepler equation, the right side getting an additional term M_1, periodic of order J_2, depending on v_0 and $\psi_s + \psi_0$. This second Kepler equation does not require a full-fledged solution, but may be solved by a differential method. Knowing E_1, one may then use (12.1) to find v_1 and (10.2) to find ψ_1.

One continues in a similar fashion to find E_2, v_2, and ψ_2. For the reference orbit the secular parts of E, v, and ψ are then known exactly and the short-periodic parts through order J_2^2. There are no long-periodic terms in this solution for the reference orbit.

Eqs. (12.1) and (12.2) then give the spheroidal coordinates ρ and η. The right ascension ϕ follows from (10.3), (14.3), and (15.3), after insertion of ψ into (12.3) to find X. This completes the solution for the reference orbit.

7. A Sketch of the Necessary Perturbation Theory

If we subtract (7) from (2), we find that the part of the gravitational potential not accounted for in the reference orbit is given by

$$V - V' = \frac{\mu r_e^3}{r^4} J_3 P_3(\sin\theta) + \frac{\mu r_e^4}{r^5} (J_4 + J_2^2)\, P_4(\sin\theta) + \dots \qquad (19)$$

As an example, we consider here only the residual fourth harmonic, so that the perturbing term in the Hamiltonian becomes

$$H' = \frac{\mu r_e^4}{r^5} (J_4 + J_2^2) P_4(\sin\theta) \tag{20}$$

If we use the formulas of elliptic motion for r and θ, viz.,

$$r = a(1-e\cos E) = \frac{a(1 - e^2)}{1 + e\cos v} \tag{21}$$

$$\sin\theta = \sin I \sin(v + \beta_2) \tag{22}$$

the perturbation H' will be correct through order J_2^2. This order of accuracy will result in secular and short-periodic changes correct through order J_2^2 and long-periodic terms correct through order J_2. (It is well to emphasize at this point that this order of accuracy is for effects produced by less than 0.5% of the departure of the planet form sphericity; for the 99.5% of this departure already accounted for by the potential (7) the secular effects are exact and long-periodic effects do not exist.)

In doing the perturbation theory, the first canonical variables that come to mind are the Jacobi "constants" α_1, α_2, α_3, β_1, β_2, and β_3. Their shortcomings are well known, however, since they lead to Poisson terms in α_1 and α_2. The next set that comes to mind is the one generated from the α's and β's by the generating function

$$S' = -\alpha_1 t + \mu(-2\alpha_1)^{-\frac{1}{2}}\beta_1' + \alpha_2\beta_2' + \alpha_3\beta_3' , \tag{23}$$

If we define n_0 by

$$\mu = n_0^2\, a_0^3 , \qquad a_0 \equiv -\mu/2\alpha_1 \tag{24}$$

this leads to the set

$$\begin{array}{ll} L = (\mu a_0)^{\frac{1}{2}} & \ell = n_0(t + \beta_1) \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 , \end{array} \tag{25}$$

canonical with respect to the Hamiltonian

$$H = -\mu^2/2L^2 + H' \tag{26}$$

One may attempt to apply the von Zeipel method in the way successfully used by Brouwer (1959), first eliminating short-periodic terms and then proceeding to eliminate long-periodic terms. One then finds, however, that the corresponding generating function S_1^* , which ought to be of the first order in the parameter

$$\sigma \equiv J_4 + J_2^2 \tag{27}$$

must satisfy

$$\frac{\partial S_1^*}{\partial \beta_1'} = \text{zeroth order in } \sigma \tag{28}$$

One may alternatively eliminate short-periodic and long-periodic terms simultaneously, but one then obtains a Poisson term of the form $v' \sin 2\beta_2'$ in $\alpha_2 - \alpha_2'$. Since v' has a secular part, such a result would appear absurd, since the "constant" α_2, which ought to have only a small periodic variation, would then become infinite. These difficulties are examples of the failure of the von Zeipel method whenever the following conditions both hold:

(1) the perturbation has a long-periodic part of the first order in the perturbation parameter σ, and (2) the canonical variables are such that the unperturbed Hamiltonian depends only on L.

The following set, however, is successful. If j_1, j_2, and j_3 are the action variables and if w_1, w_2, and w_3 are the corresponding angle variables, we define L, G, H, ℓ, g, and h by

$$2\pi L = j_1 + j_2 + j_3 \operatorname{sgn} \alpha_3 \qquad\qquad \ell = 2\pi w_1$$

$$2\pi G = j_2 + j_3 \operatorname{sgn} \alpha_3 \quad (29) \qquad\qquad g = 2\pi(w_2 - w_1) \quad (30)$$

$$2\pi H = j_3 \qquad\qquad h = 2\pi(w_3 - w_2 \operatorname{sgn} \alpha_3) ,$$

where $\operatorname{sgn} \alpha_3 = \pm 1$ respectively for a direct orbit or a retrograde orbit. To verify that they are canonical, note that

$$L d\ell + G dg + H dh = j_1 dw_1 + j_2 dw_2 + j_3 dw_3 \tag{31}$$

They were introduced by Izsak (1962) in his application of the author's theory to the problem of the critical inclination.

If

$$j_{ik} \equiv \partial j_i / \partial \alpha_k \,, \qquad (i,k = 1,2,3) \tag{32}$$

the β's are then given by

$$\begin{aligned} 2\pi(t + \beta_1) &= j_{23}(\ell + g) + j_{11}\,\ell \\ 2\pi\beta_2 &= j_{22}(\ell + g) + j_{12}\,\ell \\ 2\pi\beta_3 &= 2\pi h + j_{13}\,\ell + (j_{23} + 2\pi \operatorname{sgn} \alpha_3)(\ell + g) \end{aligned} \tag{33}$$

The constant orbital elements in the perturbed problem then become the constant parts a'', e'', and η_0'' of a, e, and η_0, along with the initial values ℓ_0'', g_0'', and h_0'' of the secular parts of ℓ, g, and h.

The corresponding Hamiltonian F is given by

$$F = F_0(L, G, H) + F_1 \tag{34}$$

where

$$F_0 = -\alpha_1 \qquad\qquad F_1 = -H' \tag{35}$$

and

$$\begin{aligned} \dot{L} &= \frac{\partial F}{\partial \ell} & \dot{\ell} &= -\frac{\partial F}{\partial L} \\ \dot{G} &= \frac{\partial F}{\partial g} & \dot{g} &= -\frac{\partial F}{\partial G} \\ \dot{H} &= \frac{\partial F}{\partial h} & \dot{h} &= -\frac{\partial F}{\partial H} \end{aligned} \tag{36}$$

One cannot express the unperturbed Hamiltonian $F_0 = -\alpha_1$ exactly as a function of L, G, and H, but it is not necessary to do so. One needs only the derivatives

$$\begin{aligned} \frac{\partial F_0}{\partial L} &= \sum_1^3 \frac{\partial F_0}{\partial J_i}\frac{\partial J_i}{\partial L} = -2\pi\nu_1 \\ \frac{\partial F_0}{\partial G} &= \sum_1^3 \frac{\partial F_0}{\partial J_i}\frac{\partial J_i}{\partial G} = 2\pi(\nu_1 - \nu_2) \\ \frac{\partial F_0}{\partial H} &= \sum_1^3 \frac{\partial F_0}{\partial J_i}\frac{\partial J_i}{\partial H} = 2\pi(\nu_2 \operatorname{sgn}\alpha_3 - \nu_3) \end{aligned} \tag{37}$$

On applying the von Zeipel method, one first carries through the simple but tedious elimination of the short-periodic terms. Proceeding to the long-periodic terms, one finds that the appropriate generating function $S_1^*(L', G'', H, g')$ must satisfy

$$\frac{\partial F_0}{\partial G''}\frac{\partial S_1^*}{\partial g'} = -(F_1)_{\text{long-periodic}}, \tag{38}$$

leading to

$$2\pi(\nu_1'' - \nu_2'')\frac{\partial S_1^*}{\partial g'} = \sigma f(L', G'', H)\cos 2g', \tag{39}$$

where f is a certain function of L', G", and H. Since $\nu_1'' - \nu_2''$ is proportional to $1\text{-}5H^2/G''^2 \approx 1\text{-}5\cos^2 I$, this leads to the familiar resonance denominator, whenever $\sigma \equiv J_4 + J_2^2 \neq 0$. Since $\sigma/(\nu_1''-\nu_2'') = O(J_2)$, the long-periodic terms are accurate through order J_2.

After finding the above canonical variables as functions of time, one easily converts their changes into changes of α_1, α_2, and α_3 or of a, e, and I. On inserting the functions a(t), e(t), I(t), ℓ(t), g(t), and h(t) into Eqs.(10), one then can find, by differential methods, the changes in E, v, ψ, and X, and thus in the coordinates, that are produced by the perturbation. It is not necessary to do a complete re-inversion of (10.1) and (10.2).

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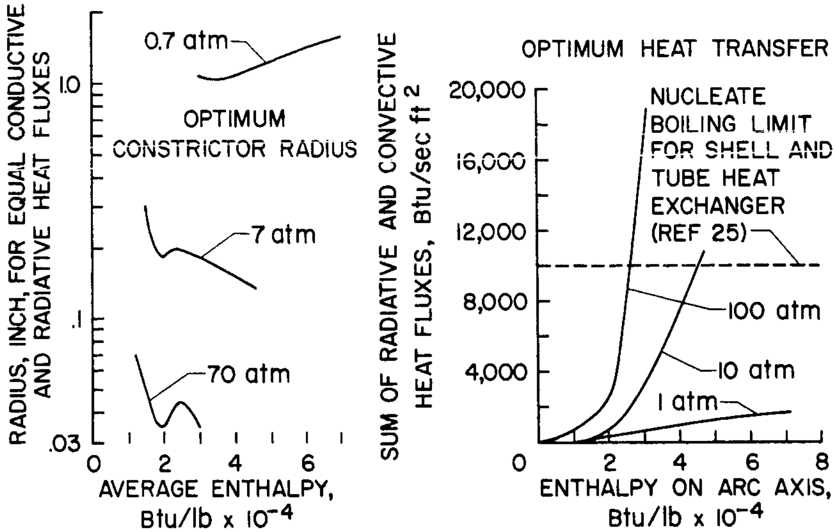

Fig. 15.- Effects of enthalpy on constrictor radius and heat transfer for equal radiative and convective heat fluxes.

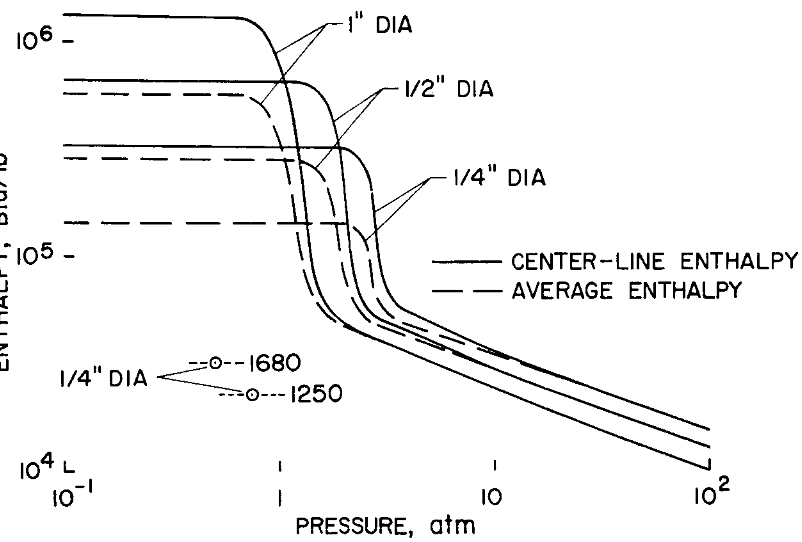

Fig. 16.- Enthalpy of the air in cylindrically constricted arcs with 10,000 Btu/sec ft^2 wall heat flux.

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